

Chapter 6: Inverse Functions

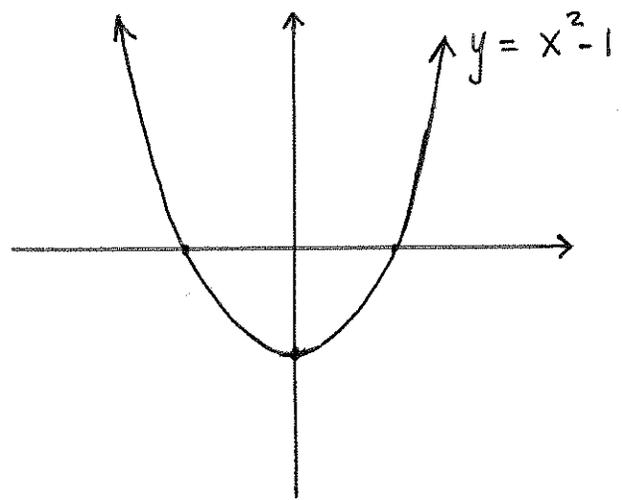
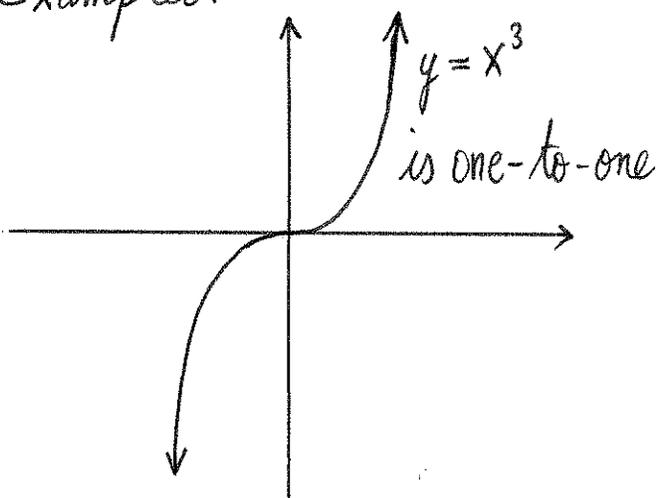
Section 6.1: Inverse Functions

Definition: A function f is called one-to-one if it never takes on the same value twice; that is

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

Graphically, f must pass the horizontal line test; that is, every y -value corresponds to only one x -value

Examples:



$$y = x^2 - 1 \text{ is not one-to-one: } (1)^2 - 1 = (-1)^2 - 1 = 0.$$

In simple terms, a function is one-to-one if each input corresponds to one output.

Examples: $f(x) = x^3$ is one-to-one, since $x_1^3 \neq x_2^3$ whenever $x_1 \neq x_2$.

$f(x) = \sin(x)$ is not one-to-one, since $\sin(x + 2k\pi) = \sin(x)$ for any integer k ; Graphically, $\sin(x)$ does not pass the horizontal line test.

Definition: Let f be one-to-one, with domain A , and Range B . Then its inverse function f^{-1} has domain B and Range A , and is defined by $f^{-1}(y) = x \iff f(x) = y$ for any y in B .

In simpler terms, if f maps x into y , then f^{-1} maps y back into x .

$$x \xrightarrow{f} y \xrightarrow{f^{-1}} x$$

Examples: Since x^3 is one-to-one, it has an inverse.

The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{\frac{1}{3}}$, because if $y = f(x) = x^3$, then $f^{-1}(y) = f^{-1}(x^3) = (x^3)^{\frac{1}{3}} = x$.

Cancellation Equations:

$$f^{-1}(f(x)) = x, \quad \text{and} \quad f(f^{-1}(x)) = x.$$

Do $f(x) = x^3$, $f^{-1}(x) = x^{\frac{1}{3}}$ as an example.

How to find the inverse of a one-to-one function f :

① write $y = f(x)$

② solve for x in terms of y

③ interchange x and y ; the resulting function is $y = f^{-1}(x)$

Example: find the inverse of $f(x) = x^3 - 2$.

(Note that $f(x) = x^3 - 2$ is indeed one-to-one, since it passes the horizontal line test)

① $y = x^3 - 2$

② $y + 2 = x^3 \Rightarrow x = \sqrt[3]{y + 2}$

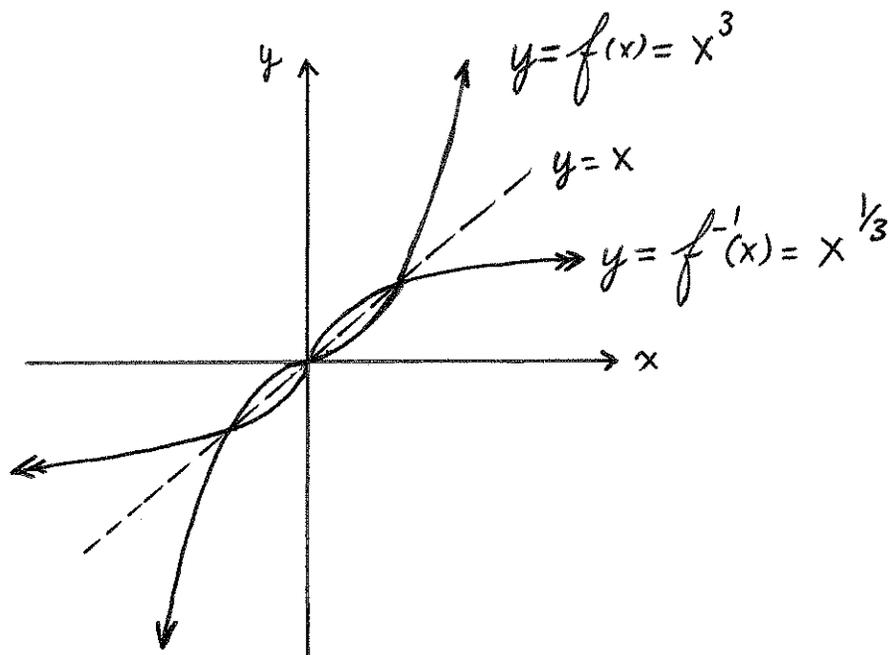
③ $f^{-1}(x) = \sqrt[3]{x + 2}$

• Observe that $f(x) = y \Leftrightarrow f^{-1}(y) = x$ means that the point (a, b) is on the graph of f , if and only if the point (b, a) is on the graph of f^{-1} .

• Graphically, the point (b, a) is the reflection of the point (a, b) about the line $y = x$.

• Therefore, the graph of f^{-1} is obtained by reflecting the graph of f about $y = x$.

Example:



Sketch of $f(x) = x^3$ and its inverse function $f^{-1}(x) = x^{1/3}$.

The Calculus of f^{-1} .

Continuity: if f is one-to-one, continuous, then so is f^{-1} .

This is because the graph of f^{-1} is simply the reflection of the graph of f about $y=x$; so if the graph of f has no breaks in it, then neither does the graph of f^{-1} .

Differentiability: Recall that $f(x)$ is differentiable at $x=a$, if $f'(a)$ exists.

Theorem: If f is one-to-one, differentiable, with inverse f^{-1} ,

Then $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$, provided $f'(f^{-1}(a)) \neq 0$.

Proof: $(f^{-1})'(a) = \lim_{x \rightarrow a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a}$

Let $f(b) = a, f'(x) = y$

So we can rewrite

$$\begin{aligned} (f^{-1})'(a) &= \lim_{y \rightarrow b} \frac{y - b}{f(y) - f(b)} = \lim_{y \rightarrow b} \frac{1}{\frac{f(y) - f(b)}{y - b}} \\ &= \frac{1}{\lim_{y \rightarrow b} \frac{f(y) - f(b)}{y - b}} = \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))} \quad \blacksquare \end{aligned}$$

Example: Restrict $y = x^2$ on the interval $[0, \infty)$.

Then $y = x^2$ is one-to-one (on this interval).

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \frac{1}{2 \cdot 2} = \frac{1}{4}.$$

Or, in this case, we could solve for $f^{-1}(x)$ explicitly.

$$y = f(x) = x^2 \rightarrow x = \sqrt{y} \rightarrow f^{-1}(x) = \sqrt{x} = x^{\frac{1}{2}}.$$

$$(f^{-1})'(x) = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}; \text{ Thus } (f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Example: Consider $f(x) = \tan(x)$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Then f is one-to-one ($f'(x) = \sec^2 x > 0 \Rightarrow f$ is increasing).

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(\frac{\pi}{4})} = \frac{1}{\sec^2(\frac{\pi}{4})} = \frac{1}{(\sqrt{2})^2} = \frac{1}{2}.$$